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Approximation Theory and Functional Equations (II)[1]

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1. INTRODUCTION

In a recent paper with a similar name, I examined the connection between the approximation of continuous functions on certain compact sets by functions of the special form $A(x) + B(y)$, and the solution of a related system of functional equations. In the present paper, I discuss a more general form of this, and illustrate it by applying it to the approximation of continuous functions of three variables by functions of the special form $A(x) + B(y) + C(z)$, or the form $A(x, y) + B(y, z) + C(z, x)$.

2. THE MAIN RESULT

Let S and T be compact spaces, and let M be any subspace of $C[S \times T]$, the space of complex-valued continuous functions on $S \times T$ with the uniform norm $\| g \| = \max | g(s, t)|$, for all $s \in S$, $t \in T$. Augment M to a subspace H by adding $C[S]$; thus, H consists of all functions on $S \times T$ of the form

$$f(s, t) = h(s) + g(s, t) \tag{1}$$

where $h \in C[S]$ and $g \in M$.

Let $\gamma_0, \gamma_1, \ldots, \gamma_n$ be a collection of continuous mappings from S into T, not necessarily 1:1, and let K be the compact set $\bigcup_0^m \gamma_j$ consisting of the union of the graphs of the γ_j. Note that $H_{|K}$ is a subspace of $C[K]$. Let Γ_{ij} be the subset of S consisting of all points $s \in S$ such that $\gamma_i(s) = \gamma_j(s)$. Finally, let $u_1, u_2, \ldots, u_n$ be continuous complex-valued functions defined on S.

Then, consider the system of functional equations

$$g(s, \gamma_{i-1}(s)) - g(s, \gamma_i(s)) = u_i(s), \qquad i = 1, 2, \ldots, n. \tag{2}$$

Given the mappings γ_i and the functions u_i, we seek conditions under which

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there exists a function $g \in M$ such that all Eqs. (2) are satisfied for all $s \in S$, or such that they are approximately satisfied with an error that is uniformly less than ϵ, for every given $\epsilon > 0$. (In the latter case, we say that (2) has approximate solutions, and in either case, we say that (2) is a solvable system.)

It is clear that (2) will not be solvable unless the functions u_i obey certain restrictions. For example, we must have $u_k(s) = 0$ for any point s where $\gamma_{k-1}(s) = \gamma_k(s)$. More generally, it is necessary that

$$u_k(s) + u_{k+1}(s) + \cdots + u_j(s) = 0 \qquad \text{for} \quad s \in \Gamma_{k-1,j} . \tag{3}$$

THEOREM 1. *The system* (2) *has an exact solution* (*approximate solutions*) $g \in M$ *for every choice of the functions* u_i *obeying the restrictions* (3) *if and only if* $H_{|K}$ *equals* $C[K]$ (*is dense in* $C[K]$).

The proof closely parallels that of the corresponding theorem in [1]. If $H_{|K} = C[K]$, consider the function F defined on K by

$$F(s, \gamma_0(s)) = 0,$$
$$F(s, \gamma_j(s)) = -\{u_1(s) + u_2(s) + \cdots + u_j(s)\},$$

for all $s \in S$ and $j = 1, 2, \ldots, n$. Then, the fact that the functions u_i satisfy restrictions (3) enables one to verify that F is continuous on K. Accordingly, there must exist a function $f \in H$ such that $f = F$ on K. Writing f as $f(s, t) = h(s) + g(s, t)$, we have

$$h(s) + g(s, \gamma_0(s)) = 0$$
$$h(s) + g(s, \gamma_j(s)) = -\{u_1(s) + \cdots + u_j(s)\}$$

for all $s \in S$. Subtracting, we see that $g \in M$ is the desired solution of the system (2). A similar argument shows that approximate solutions can be obtained if $H_{|K}$ is dense in $C[K]$.

Conversely, given any function $F \in C[K]$, consider the system of equations obtained by choosing

$$u_i(s) = F(s, \gamma_{i-1}(s)) - F(s, \gamma_i(s))$$

for all $s \in S$ and all $i = 1, 2, \ldots, n$. Note that these continuous functions obey the restrictions (3). If we assume that the system (2) is exactly solvable for every choice of the u_i obeying (3), then there must exist $g \in M$ such that

$$g(s, \gamma_{i-1}(s)) - g(s, \gamma_i(s)) = u_i(s)$$

for each i. Setting $h(s) = F(s, \gamma_0(s)) - g(s, \gamma_0(s))$, it is readily verified that the function f given by $f(s, t) = h(s) + g(s, t)$, which belongs to H, obeys $f = F$ on K.

Again, a similar argument applies if it is known that the system (2) is approximately solvable, and the conclusion is that for any $\epsilon > 0$, an $f \in H$ can be found such that $\|f - F\|_K < \epsilon$, and $H_{|K}$ is dense in $C[K]$.

3. First application

Let X, Y and Z be compact spaces, and choose S as $X \times Y$ and T as Z. Take M to be the subspace of $C[X \times Y \times Z]$ consisting of those continuous complex-valued functions of the form

$$\begin{aligned} g(s, t) &= g((x, y), z), \\ &= \varphi(x, z) + \psi(y, z), \end{aligned} \tag{4}$$

where φ and ψ are arbitrary continuous functions. Then, the space H of functions f of form (1) becomes equivalent to the space of all continuous functions on $X \times Y \times Z$ of the form

$$f(x, y, z) = A(x, y) + B(y, z) + C(z, x). \tag{5}$$

We are interested in the approximation properties of H as a subspace of $C[X \times Y \times Z]$ on special compact sets $K \subset X \times Y \times Z$.

For each $j = 0, 1, 2,..., n$, let γ_j be a continuous mapping from $X \times Y$ into Z. Let K be the compact subset of $X \times Y \times Z$ consisting of the union of the "surfaces" that are the graphs of the γ_j. Let Γ_{ij} be the subset of $X \times Y$ consisting of all the (x, y) for which $\gamma_i(x, y) = \gamma_j(x, y)$; these describe the intersections of the given surfaces.

We are now ready to apply Theorem 1, obtaining the following.

THEOREM 2. *$H_{|K}$ is dense in $C[X \times Y \times Z]$ if and only if the system of functional equations*

$$\varphi(x, \gamma_{j-1}(x, y)) - \varphi(x, \gamma_j(x, y)) + \psi(y, \gamma_{j-1}(x, y)) - \psi(y, \gamma_j(x, y)) = u_j(x, y) \tag{6}$$

admit approximate solutions φ and ψ for every choice of functions u_j that obey the restrictions

$$u_j(x, y) + u_{j+1}(x, y) + \cdots + u_k(x, y) = 0 \quad \text{on} \quad \Gamma_{j-1,k}$$

for all $j < k$.

When $X = Y = Z = [0, 1]$, smooth functions in H are those that satisfy the differential equation $\partial^3 f/\partial x\, \partial y\, \partial z = 0$ in the unit cube; H may also be characterized by the vanishing of all the functionals L of the form

$$L(f) = \sum_1^8 (-1)^{k+1} f(P_k) \tag{7}$$

where the points p_k in order are (a, b, c), (a', b, c), (a', b', c), (a, b', c), (a, b', c'), (a', b', c'), (a', b, c'), (a, b, c'). The corresponding theory parallels closely that for the simpler two variable case in which $H = \{\text{all } A(x) + B(y)\}$. H will be dense in $C[K]$, for uncomplicated K, if and only if K cannot support a linear functional which is a linear combination of point measures and which annihilates H. Such functions must have a structure similar to that in (7). We hope to return to the study of this special case in future papers.

4. Second Application

Again let X, Y and Z be compact spaces, but choose S to be X and T to be $Y \times Z$. Take M to be the subspace of $C[X \times Y \times Z]$ consisting of the functions of the form

$$\begin{aligned} g(s, t) &= g(x. (y, z)) \\ &= \varphi(y) + \psi(z) \end{aligned} \tag{8}$$

where $\varphi \in C[Y]$ and $\psi \in C[Z]$. Then the space H defined by (1) becomes the collection of all continuous functions on $X \times Y \times Z$ of the form

$$f(x, y, z) = A(x) + B(y) + C(z) \tag{9}$$

Again, we are interested in the approximation properties of these functions on (thin) compact subsets of $X \times Y \times Z$.

For each $j = 0, 1,..., n$, let γ_j be a continuous mapping from X into $Y \times Z$. Note that we can write $\gamma_j(x)$ as $(\alpha_j(x), \beta_j(x))$ where α_j and β_j are continuous mappings from X into Y and Z, respectively. The set K is the union of the graphs of the γ_j. When $X = Y = Z = [0, 1]$, the graph of each γ_j is an arc lying in the unit cube, so that K is a "thin" set. Since the set Γ_{ij} consists of all $x \in X$ where $\gamma_i(x) = \gamma_j(x)$, and since this is equivalent to both $\alpha_i(x) = \alpha_j(x)$ and $\beta_i(x) = \beta_j(x)$, these sets describe the mutual intersections of the set of arcs that comprise K.

We can now apply Theorem 1, obtaining the following.

Theorem 3. *$H_{|K}$ is dense in $C[K]$ if and only if the system of functional equations*

$$\varphi(\alpha_{j-1}(x)) - \varphi(\alpha_j(x)) + \psi(\beta_{j-1}(x)) - \psi(\beta_j(x)) = u_j(x) \tag{10}$$

admit approximate solutions φ *and* ψ *for every choice of the functions* u_j *that obey the restrictions*

$$u_j(x) + u_{j+1}(x) + \cdots + u_k(x) = 0 \qquad \textit{for} \quad x \in \Gamma_{j-1,k}$$

for all $j < k$.

Much remains to be learned about this special case. The space H is so small a subspace of $C[X \times Y \times Z]$ that its annihilator is a large class of measures on $X \times Y \times Z$, and its structure is not well understood. In particular, it is not yet clear how one should choose the curves comprising K so that K will not support nontrivial annihilating measures and so that $H_{|K}$ is dense. A first step is to investigate the behavior of the functional equations (10) when α_j and β_j are further specialized. This too may be examined in future papers.

References

1. R. C. Buck, On Approximation theory and functional equations, *J. Approximation Theory* **5** (1972), 228–237.